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ASYMPTOTIC ANALYSIS OF TORIC IDEALS

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This note is based on a lecture presented at the *Symposium on Special Differential Equations*, RIMS, Kyoto University, September 1991. It deals with recent developments in the theory of toric varieties and is related to the theory of $\mathcal{A}$-hypergeometric functions due to Gel'fand, Graev, Kapranov and Zelevinsky (see §6). With the exception of §5, which contains a new theorem on universal Gröbner bases of toric ideals, our discussion is expository and proofs are omitted. For details we refer to the articles listed in the bibliography.

1. What is "asymptotic analysis"?

Suppose we are given a geometric object $\mathcal{O}$ in complex projective space P^{n-1}. Choose any ***weight vector*** $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{Q}^n$, choose $t \in \mathbf{C}^*$, and replace each coordinate function x_i by the scaled coordinate function $t^{\omega_i} \cdot x_i$. Then we get a new deformed object $\mathcal{O}(t)$. By ***asymptotic analysis*** of $\mathcal{O}$ we mean the study of $\mathcal{O}(t)$ as t tends to infinity.

Example. *(Asymptotic analysis of hypersurfaces).* Let $\mathcal{O}$ be a hypersurface in P^{n-1}, defined by a homogeneous polynomial $f(x_1, \ldots, x_n)$ of degree p. The deformed hypersurface $\mathcal{O}(t)$ is defined by the polynomial $f(t^{\omega_1}x_1, \ldots, t^{\omega_n}x_n)$. If we view this as a univariate polynomial in t, then its leading coefficient $init_\omega(f)$ is a degree p polynomial in $\mathbf{C}[x_1, \ldots, x_n]$. We call $init_\omega(f)$ the ***initial form*** of f with respect to ω.

The distinct initial forms $init_\omega(f)$, as ω ranges over $\mathbf{Q}^n$, are in one-to-one correspondence with the faces of the *Newton polytope* $\mathcal{N}(f)$, which is the convex hull of all appearing exponent vectors. Here $init_\omega(f)$ corresponds to the face of $\mathcal{N}(f)$ which is supported by the linear functional ω. For ω sufficiently generic, the initial form $init_\omega(f)$ is a monomial, corresponding to a vertex of $\mathcal{N}(f)$.

The moduli space of all hypersurfaces of degree p in P^{n-1} is a projective space of dimension $\binom{n+p-1}{p} - 1$. The hypersurface defined by $init_\omega(f)$ equals the limit $lim_{t\to\infty}\mathcal{O}(t)$ in $P^{\binom{n+p-1}{p}-1}$. Asymptotic analysis of a hypersurface $\{f = 0\}$ therefore means the study of the closure of the torus orbit $(\mathbf{C}^*)^n \cdot f$ in $P^{\binom{n+p-1}{p}-1}$. This closure is a ***toric variety*** [13], and its corresponding lattice polytope is $\mathcal{N}(f)$.

The objects $\mathcal{O}$ to be deformed in the lecture are graded ideals $I = \oplus_{k=0}^{\infty} I_k$ in the polynomial ring $\mathbf{C}[\mathbf{x}] := \mathbf{C}[x_1, \ldots, x_n]$. The example above deals with the special case of a principal ideal $I = \langle f \rangle$. Given any graded ideal I, and given $\omega \in \mathbf{Q}^n$, we define the ***initial ideal*** $init_\omega(I)$ to be the graded ideal generated by all initial forms $init_\omega(f)$, where $f \in I$. In terms of algebraic geometry, we are interested in toric deformations of projective schemes with respect to a specific closed embedding into P^{n-1}.

By a ***projective scheme*** we mean an equivalence class $\mathcal{O}$ of graded ideals, where $I \sim I'$ if and only if $I_k = I'_k$ for $k \gg 0$. An important invariant of a projective scheme I is the ***Hilbert polynomial*** $h = h_I$ which is defined by $h(k) := dim_{\mathbf{C}}(I_k)$ for $k \gg 0$. The degree of h equals the affine dimension $dim(I)$, and its leading coefficient equals $deg(I)/dim(I)!$.

The moduli space of all projective schemes with Hilbert polynomial h in P^{n-1} is a projective scheme in some very high-dimensional projective space. It is called the ***Hilbert scheme*** and denoted $Hilb_h(P^{n-1})$. The projective scheme defined by $init_\omega(I)$ equals the limit $lim_{t \to \infty} \mathcal{O}(t)$ in $Hilb_h(P^{n-1})$. Asymptotic analysis of a graded ideal I therefore means the study of the closure of the torus orbit $(\mathbf{C}^*)^n \cdot I$ in $Hilb_h(P^{n-1})$. This closure is a toric variety; its corresponding lattice polytope is the ***state polytope*** $St(I)$ of I.

The next section contains an elementary definition of the state polytope, which is sufficient for our purposes. No familiarity with the Hilbert scheme is needed until §7.

2. Gröbner bases

A finite generating set $\mathcal{G}$ of a graded ideal $I \subset \mathbf{C}[\mathbf{x}]$ is a ***Gröbner basis*** with respect to $\omega \in \mathbf{Q}^n$ if the initial ideal $init_\omega(I)$ is generated by the set of initial forms $\{init_\omega(g) : g \in \mathcal{G}\}$. If ω has the property that $init_\omega(I)$ is a monomial ideal, then we call ω a ***term order*** for I. There is a well-known algorithm due to Buchberger for computing a Gröbner basis $\mathcal{G}$ from any generating set for I.

A monomial $\mathbf{x}^\alpha$, $\alpha \in \mathbf{N}^n$, is called ***standard*** if $\mathbf{x}^\alpha \notin init_\omega(I)$. A pair $(\mathbf{x}^\alpha, \tau)$ consisting of a monomial $\mathbf{x}^\alpha$ and a subset of variables $\tau \subset \{x_1, \ldots, x_n\}$ is a ***standard pair*** if τ has cardinality $dim(I)$, no variable in τ appears in $\mathbf{x}^\alpha$, and <u>all</u> monomials in $\mathbf{x}^\alpha \cdot \mathbf{C}[\tau]$ are standard. Here are a few basic facts from Gröbner basis theory:

- The set of standard monomials is a $\mathbf{C}$-vector space basis for the residue ring $\mathbf{C}[\mathbf{x}]/I$. The unique expansion of a polynomial f (mod I) in terms of standard monomials can be computed using normal form reduction (see [4]) with respect to a Gröbner basis $\mathcal{G}$.
- The number of standard pairs equals the degree of I.
- There exists a finite ***universal Gröbner basis*** $\mathcal{U}$ for I. This means that $\mathcal{U}$ is a Gröbner basis for I with respect to <u>all</u> term orders $\omega \in \mathbf{Q}^d$ simultaneously.

We call two weight vectors $\omega, \omega' \in \mathbf{Q}^d$ ***equivalent*** if $init_\omega(I) = init_{\omega'}(I)$. The equivalence classes are relatively open polyhedral cones, defining the ***Gröbner fan*** of I. The Gröbner fan is the normal fan of a polytope in $\mathbf{Q}^n$. Any such polytope will be denoted $St(I)$ and called a ***state polytope*** of I. For details on Gröbner fans and state polytopes as well as many references to Gröbner basis theory see [1],[12],[14].

A Gröbner basis computation for an ideal I can be interpreted geometrically as deforming the projective scheme I to the initial scheme $init_\omega(I)$. If ω is a term order, then $init_\omega(I)$ is supported on a union of coordinate flats in P^{n-1}.

3. Example: The twisted cubic curve

Let I denote the kernel of the ring map

$$\mathbf{C}[a,b,c,d] \rightarrow \mathbf{C}[x,y]\,,\; a \mapsto x,\; b \mapsto xy,\; c \mapsto xy^2,\; d \mapsto xy^3.$$

The projective variety of I is an irreducible curve in P^3, called the ***twisted cubic curve.*** The following set of five polynomials is a universal Gröbner basis for I:

$$\mathcal{U} \quad = \quad \{\, ac - b^2,\; bd - c^2,\; ad - bc,\; a^2d - b^3,\; ad^2 - c^3 \,\}.$$

Note that the first three polynomials suffice to generate the ideal I. In the following table we list all initial ideals of I and their primary decompositions.

Term order ω	initial ideal	primary decomposition	underlying cycle
$(0,0,1,3)$	$\langle ad, bd, ac\rangle$	$\langle a,b\rangle \cap \langle a,d\rangle \cap \langle c,d\rangle$	$[ab][bc][cd]$
$(0,0,-4,-3)$	$\langle ad, b^2, bd\rangle$	$\langle a,b\rangle \cap \langle b^2 d\rangle$	$[ac]^2[cd]$
$(0,0,-4,-5)$	$\langle ad^2, b^2, bc, bd\rangle$	$\langle a,b\rangle \cap \langle b,d^2\rangle \cap \langle b^2,c,d\rangle$	$[ac]^2[cd]$
$(0,0,-4,-7)$	$\langle b^2, bc, bd, c^3\rangle$	$\langle b,c^3\rangle \cap \langle b^2,c,d\rangle$	$[ac]^3$
$(0,0,-4,-9)$	$\langle b^2, bc, c^2\rangle$	$\langle b^2, bc, c^2\rangle$	$[ac]^3$
$(0,0,1,-1)$	$\langle ac, b^3, bc, c^2\rangle$	$\langle b^3,c\rangle \cap \langle a,b,c^2\rangle$	$[ac]^3$
$(0,0,2,1)$	$\langle a^2d, ac, bc, c^2\rangle$	$\langle a^2,c\rangle \cap \langle c,d\rangle \cap \langle a,b,c^2\rangle$	$[ab][bd]^2$
$(0,0,2,3)$	$\langle ac, ad, c^2\rangle$	$\langle a,c^2\rangle \cap \langle c,d\rangle$	$[ab][bd]^2$

In the last column we list the algebraic cycle underlying each initial ideal. By [10], the distinct initial cycles appear as the extreme terms in the Chow form of I, which equals

$$\underline{[ab][bc][cd]} \;-\; \underline{[ab][bd]^2} \;-\; \underline{[ac]^2[cd]} \;-\; \underline{[ad]^3} + 3[ab][ad][cd] \;+\; [ad]^2[bc].$$

Thus the scheme-theoretic analysis is strictly finer than the cycle-theoretic analysis (cf. §7).

The state polytope $St(I)$ of the twisted cubic is a planar octagon. We depict it together with its normal fan, the Gröbner fan of I:

4. Gröbner bases of toric ideals

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a subset of $\mathbf{Z}^d$ which spans an affine hyperplane in $\mathbf{Q}^d$. Then the monoid $\mathcal{M}(\mathcal{A})$ generated by $\mathcal{A}$ is graded and has maximal rank d. Its monoid algebra $\mathbf{C}[\mathcal{A}]$ is the subalgebra of $\mathbf{C}[y_1, \ldots, y_d, y_1^{-1}, \ldots, y_d^{-1}]$ generated by the Laurant monomials $\mathbf{y}^{a_1}, \ldots, \mathbf{y}^{a_n}$. The canonical epimorphism of monoids

$$\pi \, : \, \mathbf{N}^n \, \rightarrow \, \mathcal{M}(\mathcal{A}) \, , \ \lambda = (\lambda_1, \ldots, \lambda_n) \, \mapsto \, \lambda_1 a_1 + \ldots + \lambda_n a_n$$

extends to an epimorphism of monoid algebras

$$\hat{\pi} \, : \, \mathbf{C}[\mathbf{x}] \, \rightarrow \, \mathbf{C}[\mathcal{A}] \, , \ x_i \, \mapsto \, \mathbf{y}^{a_i}.$$

The kernel of $\hat{\pi}$ is denoted $\mathcal{I}_{\mathcal{A}}$ and called the *toric ideal* of $\mathcal{A}$. Its projective variety is a $(d-1)$-dimensional toric variety in P^{n-1}. The degree of $\mathcal{I}_{\mathcal{A}}$ equals the normalized volume $vol(Q)$ of the $(d-1)$-polytope $Q = conv(\mathcal{A})$.

Lemma. *The toric ideal $I_{\mathcal{A}}$ is generated by* $\{\, \mathbf{x}^\lambda - \mathbf{x}^\mu \, : \, \lambda, \mu \in \mathbf{N}^n \, , \, \pi(\lambda) = \pi(\mu) \,\}$.

A *section* of the monoid epimorphism π is a map $\sigma : \mathcal{M}(\mathcal{A}) \rightarrow \mathbf{N}^n$ such that the composition $\pi \circ \sigma$ is the identity on $\mathcal{M}(\mathcal{A})$. A section σ is called a *regular triangulation* of $\mathcal{M}(\mathcal{A})$ if there exists $\omega \in \mathbf{Q}^n$ such that $\langle \omega, \sigma(\alpha) \rangle \, < \, \langle \omega, \lambda \rangle$ for all $\alpha \in \mathcal{M}(\mathcal{A})$, $\lambda \in \pi^{-1}(\alpha) \setminus \{\sigma(\alpha)\}$. Note that every generic $\omega \in \mathbf{Q}^n$ defines a regular triangulation $\sigma = \sigma_\omega$.

Theorem. [15, §3.3] *There is a one-to-one correspondence $\sigma_\omega \mapsto init_\omega(I_{\mathcal{A}})$ between the regular triangulations of the monoid $\mathcal{M}(\mathcal{A})$ and the initial ideals of the toric ideal $I_{\mathcal{A}}$.*

Corollaries. Let $\omega \in \mathbf{Q}^n$ be a term order for $I_{\mathcal{A}}$, let $\mathcal{G}$ be a Gröbner basis, and let $\sigma : \mathcal{M}(\mathcal{A}) \rightarrow \mathbf{N}^n$ be the corresponding regular triangulation of $\mathcal{M}(\mathcal{A})$.

(a) The image of σ consists precisely of the (exponent vectors of) standard monomials.
(b) The normal form of $\mathbf{x}^\lambda$ modulo $\mathcal{G}$ equals $\mathbf{x}^\nu$ where $\nu = \sigma(\pi(\lambda))$.
(c) The number of standard pairs $(\mathbf{x}^\nu, \tau)$ equals $vol(Q)$, where $Q = conv(\mathcal{A})$.
(d) The set $\{\, \tau \, : \, (1, \tau) \text{ standard pair} \,\}$ defines a triangulation Δ_ω of the polytope Q. (Note that if $(\mathbf{x}^\nu, \tau)$ is standard then $(1, \tau)$ is standard.)

Statement (d) implies that the *secondary polytope* $\Sigma(\mathcal{A})$ is a Minkowski summand of the state polytope $St(I_{\mathcal{A}})$. This result can be sharpened using the following geometric construction. For $\alpha \in \mathcal{M}(\mathcal{A})$ consider the lattice polytope $conv(\pi^{-1}(\alpha))$ in $\mathbf{Q}^n$. We define

$$\Sigma(\mathcal{A}) \quad := \quad \int_{x \in \mathcal{M}(\mathcal{A})} conv(\pi^{-1}(\alpha)) \, d\alpha.$$

Here "$d\alpha$" can be any measure on $\mathcal{M}(\mathcal{A})$ which is supported on a sufficiently large but finite set. Then $\Sigma(\mathcal{A})$ is a polytope (called the *monoid fiber polytope*) whose normal fan is the common refinement of the normal fans of $conv(\pi^{-1}(\alpha))$ where α runs over $\mathcal{M}(\mathcal{A})$.

Theorem. *The state polytope $St(I_{\mathcal{A}})$ equals the monoid fiber polytope $\Sigma(\mathcal{A})$.*

5. Unimodular configurations

The configuration $\mathcal{A} \subset \mathbf{Z}^d$ is called *unimodular* if, for each subset $\mathcal{A}' \subseteq \mathcal{A}$, the quotient $span_{\mathbf{Z}}(\mathcal{A})/span_{\mathbf{Z}}(\mathcal{A}')$ is a free abelian group. Equivalently, all maximal minors of a matrix representing π are equal to $-1, 0$ or $+1$. An irreducible polynomial $\mathbf{x}^\lambda - \mathbf{x}^\mu$ in the toric ideal $\mathcal{I}_\mathcal{A}$ is a *circuit of* $\mathcal{A}$ if the set of variables appearing in $\mathbf{x}^\lambda - \mathbf{x}^\mu$ is minimal with respect to inclusion. (These definitions are consistent with the usage in *matroid theory*).

For general configurations $\mathcal{A}$, the circuits define the toric variety set-theoretically, but they do not generate the toric ideal $I_\mathcal{A}$. For instance, in the case of the twisted cubic (see §3) there are four circuits $ac - b^2$, $bd - c^2$, $a^2d - b^3$, $ad^2 - c^3$. They do not generate $I_\mathcal{A}$.

For unimodular configurations we have the following result, which is new.

Theorem. *If $\mathcal{A}$ is unimodular, then the set of circuits is a universal Gröbner basis for the toric ideal $I_\mathcal{A}$, and the state polytope $St(I_\mathcal{A})$ coincides with the secondary polytope $\Sigma(\mathcal{A})$.*

Proof: Let $\mathcal{U} \subset I_\mathcal{A}$ be the set of all circuits, and suppose that $\mathcal{U}$ is not a universal Gröbner basis. Then there exists a term order $\omega \in \mathbf{Q}^n$ and a monomial $\mathbf{x}^\alpha \in init_\omega(I_\mathcal{A})$ which does not lie in $\langle init_\omega(\mathcal{U})\rangle$. We may choose $\mathbf{x}^\alpha$ to have minimal total degree with this property. The monomial $\mathbf{x}^\alpha$ is the initial term of a polynomial of the form $\mathbf{x}^\beta(\mathbf{x}^\gamma - \mathbf{x}^\delta) \in I_\mathcal{A}$, say $\alpha = \beta + \gamma$, such that $\gamma, \delta \in \mathbf{N}^n$ have disjoint supports $supp(\gamma), supp(\delta) \subset \{1, \ldots, n\}$.

Using standard arguments of oriented matroid theory [3], we can choose a circuit $\mathbf{x}^\lambda - \mathbf{x}^\mu \in \mathcal{U}$ with $supp(\lambda) \subseteq supp(\gamma)$ and $supp(\mu) \subseteq supp(\delta)$. By unimodularity, all coordinates of λ and μ are either 0 or 1. Hence $\mathbf{x}^\lambda$ divides $\mathbf{x}^\gamma$, and $\mathbf{x}^\mu$ divides $\mathbf{x}^\delta$.

Since $\mathbf{x}^\alpha$ does not lie in $\langle init_\omega(\mathcal{U})\rangle$, none of its factors does. In particular, neither $\mathbf{x}^\lambda$ nor $\mathbf{x}^{\gamma-\lambda}$ lies in $\langle init_\omega(\mathcal{U})\rangle$, and therefore $\mathbf{x}^\mu$ is the initial term of $\mathbf{x}^\lambda - \mathbf{x}^\mu$. But then $\mathbf{x}^{\gamma-\lambda}$ is the initial term of $\mathbf{x}^{\gamma-\lambda} - \mathbf{x}^{\delta-\mu} \in I_\mathcal{A}$. Hence $\mathbf{x}^{\gamma-\lambda}$ is a monomial in $init_\omega(I_\mathcal{A}) \backslash \langle init_\omega(\mathcal{U})\rangle$ whose total degree is strictly less than the total degree of $\mathbf{x}^\alpha$. This contradicts our choice and hence completes the proof. $\triangleleft$.

We briefly mention an application to determinantal ideals (cf. [14, §6],[17]). Consider the product of standard simplices $\Delta_{m-1} \times \Delta_{n-1}$, and let $\mathcal{A} \subset \mathbf{Z}^{m+n}$ be its set of vertices. The elements of $\mathcal{A}$ are indexed by the elements of a generic $m \times n$-matrix (x_{ij}). The toric ideal $I_\mathcal{A}$ is the ideal in $\mathbf{C}[x_{ij}]$ which is generated by all 2×2-minors of (x_{ij}). The toric variety of $I_\mathcal{A}$ is the product of projective spaces $P^{m-1} \times P^{n-1}$ in its Segre embedding.

The matroid of the configuration $\mathcal{A}$ is the graphic matroid of the complete bipartite graph $K_{m,n}$. It is known that graphic matroids are unimodular, and hence the configuration $\mathcal{A}$ is unimodular. Here the set of circuits equals

$$\mathcal{U} \;:=\; \big\{\, x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_p j_p} - x_{i_2 j_1} x_{i_3 j_2} \cdots x_{i_1 j_p} \;:\; 1 \le i_1, \ldots, i_p \le m, 1 \le j_1, \ldots, j_p \le n \,\big\}.$$

Corollary. *The set $\mathcal{U}$ is a universal Gröbner basis for the 2×2-determinantal ideal $I_\mathcal{A}$.*

6. A-hypergeometric functions

We recall the system of $\mathcal{A}$-hypergeometric differential equations which was introduced by Gel'fand, Graev, Kapranov and Zelevinsky. Consider the toric ideal $I_{\mathcal{A}}$ in the polynomial ring of differential operators $\mathbf{C}[\partial_1, \ldots, \partial_n]$. A function $\phi(x_1, \ldots, x_n)$ on an open subset of $\mathbf{C}^n$ is called *$\mathcal{A}$-hypergeometric* with parameters $\gamma \in \mathbf{C}^d$ provided

(a) *ϕ is annihilated by the toric differential ideal $I_{\mathcal{A}}$, and*

(b) *ϕ satisfies the homogeneity condition $\phi(y^{a_1}x_1, \ldots, y^{a_n}x_n) = y^\gamma \cdot \phi(x_1, \ldots, x_n)$.*

The requirement (b) is equivalent to the statement that ϕ is annihilated by the linear differential operators $\sum_{i=1}^n a_{ij}x_i\partial_i - \gamma_j$, where $j = 1, 2, \ldots, d$. We note that the $\mathcal{A}$-hypergeometric functions defined by the 2×2-determinantal ideal in §5 are the *Aomoto-Gelfand functions*, expressed in local coordinates on the Grassmannian $Gr_m(\mathbf{C}^{m+n})$.

The main results in [6],[7],[8] are based on an asymptotic analysis of the holonomic system (a) & (b). In this section we sketch a slight refinement of this analysis, using Gröbner bases of toric ideals. The potential applications of our methods include also the p-adic theory of generalized hypergeometric functions due to Dwork [5].

Choose any term order $\omega \in \mathbf{Z}^d$ and replace x_i by $t_i^\omega x_i$ and ∂_i by $t^{-\omega_i}\partial_i$ respectively. If t tends to infinity, then the linear equations (b) remain unchanged, while the non-linear equations (a) are replaced by

(a') *ϕ is annihilated by the initial ideal $init_\omega(I_{\mathcal{A}})$ of the toric differential ideal.*

The $\mathbf{C}$-vector space of all polynomial solutions to the "model system" (a') is spanned by the set of standard monomials. In order to construct all formal solutions to the system (a') & (b), we proceed as follows. For each standard pair (x^α, τ), $\alpha \in \mathbf{N}^n, \tau \subset \{1, \ldots, n\}$, we define the formal monomial $m_{\alpha,\tau} := x^\alpha x^\beta$, where the vector $\beta \in \mathbf{C}^n$ is uniquely determined by the conditions $\pi(\beta) = \gamma - \pi(\alpha)$ and $supp(\beta) \subseteq \tau$.

Proposition. *In an open subset of $\mathbf{C}^n$, the $\mathbf{C}$-vector space of solutions to the system (a') & (b) has the basis of monomials $\{m_{\alpha,\tau}\}$ which is indexed by all standard pairs.*

This implies in particular that the system (a') & (b) is holonomic of rank $vol(Q) = degree(I_{\mathcal{A}})$. Note that the index sets τ appearing in the basis $\{m_{\alpha,\tau}\}$ are precisely the maximal simplices in the regular triangulation of Q defined by ω.

This proposition can now be used to refine the construction of Gel'fand-Kapranov-Zelevinsky in [8]: Each monomial $m_{\alpha,\tau}$ is the initial term of a unique Γ-series $\phi_{\alpha,\tau}$ which satisfies (a) & (b) and which converges in an open subset of $\mathbf{C}^n$. The set $\{\phi_{\alpha,\tau}\}$, indexed by all standard pairs, is a basis for space of $\mathcal{A}$-hypergeometric functions in that open subset. We illustrate this construction for the example in §3.

Example. *($\mathcal{A}$-hypergeometric system of the twisted cubic)*

Let $\mathcal{A} = \{(1,0), (1,1), (1,2), (1,3)\}$, let $\gamma = (\gamma_1, \gamma_2) \in \mathbf{C}^2$, and choose the term order

$\omega = (3,2,0,7) \in \mathbf{Z}^4$. The toric ideal equals $I_{\mathcal{A}} = \langle \underline{\partial_1\partial_4} - \partial_2\partial_3,\ \underline{\partial_2^2} - \partial_1\partial_3,\ \underline{\partial_2\partial_4} - \partial_3^2 \rangle$, where the initial terms are underlined. The initial ideal has the primary decomposition $init_\omega(I_{\mathcal{A}}) = \langle \partial_2^2, \partial_4 \rangle \cap \langle \partial_1, \partial_2 \rangle$. We read off the following set of three standard pairs: $(1, \{x_1, x_3\})$, $(x_2, \{x_1, x_3\})$, $(1, \{x_3, x_4\})$. This shows that the deformed system (a') & (b) is holonomic of rank 3, and its solution space has the basis

$$\left\{ m_{1,13} = x_1^{\gamma_1-\gamma_2/2} x_3^{\gamma_2/2},\ m_{x_2,13} = x_2 x_1^{\gamma_1-(\gamma_2-1)/2} x_3^{(\gamma_2-1)/2},\ m_{1,34} = x_3^{3\gamma_1-\gamma_2} x_4^{\gamma_2-2\gamma_1} \right\}$$

We lift this to a basis of Γ-series for the $\mathcal{A}$-hypergeometric system (a) & (b) as follows:

$$\begin{aligned}
\phi_{1,13} &:= \sum_{(i_1,i_2,i_3,i_4)\in\mathcal{L}_{\mathcal{A}}} \frac{x_1^{\gamma_1-\gamma_2/2+i_1} x_2^{i_2} x_3^{\gamma_2/2+i_3} x_4^{i_4}}{\Gamma(\gamma_1-\gamma_2/2+i_1+1)\Gamma(i_2+1)\Gamma(\gamma_2/2+i_3+1)\Gamma(i_4+1)} \\
\phi_{x_2,13} &:= \sum_{(i_1,i_2,i_3,i_4)\in\mathcal{L}_{\mathcal{A}}} \frac{x_1^{\gamma_1-(\gamma_2-1)/2+i_1} x_2^{i_2+1} x_3^{(\gamma_2-1)/2+i_3} x_4^{i_4}}{\Gamma(\gamma_1-\frac{\gamma_2-1}{2}+i_1+1)\Gamma(i_2+2)\Gamma(\frac{\gamma_2-1}{2}+i_3+1)\Gamma(i_4+1)} \\
\phi_{1,34} &:= \sum_{(i_1,i_2,i_3,i_4)\in\mathcal{L}_{\mathcal{A}}} \frac{x_1^{i_1} x_2^{i_2} x_3^{3\gamma_1-\gamma_2+i_3} x_4^{\gamma_2-2\gamma_1+i_4}}{\Gamma(i_1+1)\Gamma(i_2+1)\Gamma(3\gamma_1-\gamma_2+i_3+1)\Gamma(\gamma_2-2\gamma_1+i_4+1)}
\end{aligned}$$

Here $\mathcal{L}_{\mathcal{A}} := \{ (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \in \mathbf{Z}^4 : \lambda_1(1,0)+\lambda_2(1,1)+\lambda_3(1,2)+\lambda_4(1,3) = (0,0) \}$.

7. Asymptotic analysis of projective varieties

Our results on toric ideals are part of a broader research program in geometry, whose objective is the study of polyhedral invariants of projective varieties. (See the articles listed below.) This includes the study of state polytopes, Chow polytopes, secondary and fiber polytopes, etc... In this section we explain the connections between these polytopes.

The asymptotic analysis of an irreducible subvariety X of P^{n-1} deals with a compactification of the orbit of X under the torus $(\mathbf{C}^*)^n$. In order to obtain such a compactification, we need to choose a suitable moduli space. There are two natural choices:

(1) The *Chow variety* which parametrizes all algebraic cycles on P^{n-1} of a fixed degree;
(2) the *Hilbert scheme* of all subschemes of P^{n-1} having a fixed Hilbert polynomial.

In both cases the ambient space P^{n-1} can be replaced by a product of projective spaces, or, more generally, by a projective toric variety X_P. Both theories (1) and (2) can thus be extended to relative theories (1-rel) and (2-rel), which deal with the *relative Chow variety* or the *relative Hilbert scheme* of subvarieties X which are embedded in X_P.

In each of these four theories we can now restrict to the important special case where X itself is toric. This leads to four subtheories (1-tor), (2-tor), (1-rel-tor), (2-rel-tor). So, in total there are eight theories dealing with the asymptotic analysis of projective varieties. For each of these theories we need specific ***algebraic tools*** and we obtain specific *polytopes* which encode the toric deformations of X:

(1) The algebraic tool for encoding the point of X on the Chow variety is the ***Chow form*** R_X. The corresponding polytope is the ***Chow polytope*** $Ch(X)$ [10].

(1-tor) If X itself is toric, defined by a toric ideal $I_{\mathcal{A}}$, then the Chow form is the ***$\mathcal{A}$-resultant*** $\mathcal{R}_{\mathcal{A}}$. Here the Chow polytope coincides with the ***secondary polytope*** $\Sigma(\mathcal{A})$ [10, §5].

(1-rel) The toric deformations of a subvariety X of a toric variety X_P are encoded by the ***relative Chow polytope*** $Ch(X_P, X)$. This polytope has not yet been studied in detail.

(1-tor-rel) If X is a toric subvariety of a toric variety X_P, then we have a projection of polytopes $P \to Q$, and the relative Chow polytope equals the ***fiber polytope*** $\Sigma(P, Q)$ [2],[11].

(2) Gröbner bases are the main algebraic tool for describing the point of a graded ideal I on the Hilbert scheme. The corresponding polytope is the ***state polytope*** $St(I)$, whose vertices correspond to distinct initial ideals of I [1],[12].

(2-tor) The state polytope of a toric ideal $I_{\mathcal{A}}$ is the monoid fiber polytope $St(I_{\mathcal{A}})$, whose vertices correspond to regular triangulations of $\mathcal{M}(\mathcal{A})$. See §4 and [14], [15,§3.3].

(2-rel) In a forthcoming paper we plan to develop a ***relative Gröbner basis theory*** for subschemes I of a toric variety X_P. Here we get the ***relative state polytope*** $St(X_P, I)$ whose vertices correspond to initial ideals of I in the monoid algebra of X_P.

(2-rel-tor) The relative state polytope for a toric subscheme $I_{\mathcal{A}}$ of a toric variety X_P is the monoid fiber polytope $\Sigma(\mathcal{B}, \mathcal{A})$ as defined in [16, §7]. This is a refinement of the concept of fiber polytopes, which has interesting potential applications in integer programming.

The natural morphism from the Hilbert scheme to the Chow variety implies that the polytope in the theory (1-$*$) is always a Minkowski summand of the polytope in the theory (2-$*$). For instance, it was shown in [10, §3] that the Chow polytope $Ch(X)$ is a Minkowski summand of the state polytope $St(I)$ if I is the vanishing ideal of X. Likewise, the fiber polytope $\Sigma(P, Q)$ is a Minkowski summand of the monoid fiber polytope $\Sigma(\mathcal{B}, \mathcal{A})$ if $P = conv(\mathcal{B})$ and $Q = conv(\mathcal{A})$ as in [16].

On the other hand, the polytope in the relative theory ($*$-rel) is always the projection of the polytope in ($*$). For instance, the fiber polytope $\Sigma(P, Q)$ is a projection of the secondary polytope $\Sigma(Q) = \Sigma(\mathcal{A})$. Likewise, the monoid fiber polytope $\Sigma(\mathcal{B}, \mathcal{A})$ is a projection of the state polytope $St(I_{\mathcal{A}})$ of the toric ideal.

In view of our discussion in §6, it would be interesting to extend this picture by developing theories (3-$*$) for the asymptotic analysis of holonomic systems of differential equations. Here (3-tor) would concern deformations of $\mathcal{A}$-hypergeometric functions and (3-rel-tor) would concern $\mathcal{A}$-hypergeometric functions defined on a toric variety X_P.

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